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Neveu-Schwarz sheaves and differential equations for Mumford superforms

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Dedicated to I.M. Gelfand on his 75th birthday

Abstract. *Multiloop contributions in the Polyakov formulation of the string (resp. superstring) theory are calculated via a measure on the moduli space of curves (resp. supercurves) which equals the modulus squared of the Mumford form (resp. superform). In* [2] *it is shown that the Mumford form is a horizontal section of a canonical connection. In this paper we extend this proof to superforms.*

INTRODUCTION

The comparison of path integral and operator quantization in the two-dimensional conformal field theory and in the quantum theory of bosonic and fermionic strings led to the discovery of unexpected ties between representations of the Virasoro algebra and moduli spaces of curves (perhaps, endowed with additional structures); cf. [1, 2, 6, 7] among others.

Roughly speaking, the overall picture includes at least the following constructions.

a) A Virasoro algebra can be defined on any Riemann surface as a central

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extension of the algebra (or, rather, sheaf of Lie algebras) of vector fields. The center is the de Rham sheaf, i.e., a central charge is associated with any point and any closed curve on the surface: cf. [1].

b) The usual Virasoro algebra acts upon the moduli space of triples: (a Riemann surface, a point on it, a formal parameter at this point): cf. [2, 6, 7]. Unlike the case a), this action changes the surface itself, i.e. nontrivially projects upon the moduli space of Riemann surfaces. This projection can be used to write differential equations for the Polyakov measure, for the correlation functions, etc. In [1], a connection between a) and b) was conjecturally stated. In [2], this was proved for the bosonic case using a different version of construction a).

In this paper we extend this result to super-Riemann surfaces. We hope to return elsewhere to fermionic strings in connection with the following result, sketched in [2]:

c) The representations of the Virasoro algebra belonging to the «discrete series» give rise (with the help of b)) to some fait connections on vector bundles over moduli spaces and thus to representations of (the central extension of) the Teichmüller group.

I am happy to dedicate this paper to I.M. Gelfand. His philosophy of representation theory and formal geometry deeply influenced the subject discussed here.

This paper may be considered as a supplement to (the first three sections of) [2]. I have stressed special features of supercurves but omitted some calculations similar to those of [2].

1. PRELIMINARIES ON SUPERCURVES

1.1. **SUSY-Curves.** Let $\pi : X \to S$ be a SUSY-family (cf. [3], $N = 1$), i.e. a family of Riemannian supersurfaces parametrized by a complex analytic supermanifold S. Recall that a relative local coordinate system $Z = (z, \zeta)$ on X is called compatible (with the given SUSY-structure) if $D_Z = \dfrac{\partial}{\partial \zeta} + \zeta \dfrac{\partial}{\partial z}$ generates the structural subsheaf $T^1_{X/S}$ of the relative tangent sheaf $T_{X/S}$. We denote by $\omega_{X/S}$ the dual sheaf $(T^1_{X/S})^*$ of rank $0 \mid 1$. This is canonically isomorphic to $\mathrm{Ber}\, \Omega^1_{X/S}$ and is a relative dualizing sheaf. Denote by dZ a local section of $\omega_{X/S}$ for which $(dZ, D_Z) = 1$ and define the differential operator

$$\delta : O_X \to \omega_{X/S} = \omega$$

by the local formula $\delta f = dZ \cdot D_Z f$. This is clearly independent of the choice of a compatible coordinate system Z and defines the right version of the relative de Rham complex on X/S.

Let $\sigma : S \to X$ be a S-point of X, i.e. a section of π. The image of σ is a closed subspace of X of codimension $1 \mid 1$: points of a SUSY-curve are not divisors. However, each point can be canonically embedded into a divisor with the same support. Namely, let Z be a compatible coordinate and $z - z_0 = \zeta - \zeta_0 = 0$ be the local equations of the point. Then the divisor $z - z_0 - \zeta\zeta_0 = 0$ does not depend on the choice of Z and hence is defined globally over S. We shall prove this in a universal setting, defining the relative «superdiagonal» $\Delta^S \subset X \underset{S}{\times} X$.

1.2. Superdiagonal. Let $J \subset O_{X \underset{S}{\times} X}$ be the sheaf defining the relative diagonal $i : \Delta \to X \underset{S}{\times} X$. We have the usual exact sequence for $O_{\Delta(1)} = O_{X \underset{S}{\times} X}/J^2$

$$0 \to J/J^2 \to O_{\Delta(1)} \to O_\Delta \to 0.$$

Furthermore, $J/J^2 = i_*(\Omega^1_{X/S})$. Define $\tilde{\delta} : \Omega^1_{X/S} \to \omega_{X/S}$ as an O_X-linear map for which $\tilde{\delta}(df) = \delta f$. Put $I = \operatorname{Ker} \tilde{\delta}$ and $O_{\Delta^S} = O_{\Delta(1)}/I$. We shall often identify sheaves on $X_S \times X$ supported on the diagonal with sheaves on X.

1.3. Lemma. *Δ^S is a closed analytic subspace of codimension $1 \mid 0$, called the (relative) superdiagonal of X. For a compatible system $Z = (z, \zeta)$ put $Z_i = (z_i, \zeta_i) = p_i^*(Z)$, where $p_{1,2} : X \underset{S}{\times} X \to X$ are projections. Then Δ^S is defined by the equation $z_2 - z_1 - \zeta_2\zeta_1 = 0$.*

Proof. Under a standard identification

$$z_2 - z_1 - \zeta_2\zeta_1 \bmod J^2 = dz - d\zeta \cdot \zeta.$$

Moreover, $\tilde{\delta}(dz - d\zeta \cdot \zeta) = D_Z z - D_Z \zeta \cdot \zeta = 0$. Hence $(z_2 - z_1 - \zeta_2\zeta_1) \bmod J^2 \in I$. One easily checks that in fact locally $I = (z_2 - z_1 - \zeta_2\zeta_1) + J^2$. It remains to check that $J^2 \subset (z_2 - z_1 - \zeta_2\zeta_1)$. In fact, $J^2 = ((z_2 - z_1)^2, (z_2 - z_1)(\zeta_2 - \zeta_1))$, and we have

$$(z_2 - z_1)^2 = (z_2 - z_1 - \zeta_2\zeta_1)(z_2 - z_1 + \zeta_2\zeta_1),$$
$$(z_2 - z_1)(\zeta_2 - \zeta_1) = (z_2 - z_1 - \zeta_2\zeta_1)(\zeta_2 - \zeta_1). \qquad \blacksquare$$

We shall now define a superresidue map ress. We start with a formal situation. Let A be a supercommutative ring of constants.

1.4. Lemma. *Let $\delta : A((z, \zeta)) = O \to dZ \cdot A((z, \zeta)) = \omega$ be the map $\delta f =$*

$= dZ \cdot D_Z f$. *Define* ress: $\omega \to A$ *as a continuous A-linear map with*

$$\operatorname{ress}(dZ \cdot z^a \zeta^b) = \begin{cases} 1 \text{ for } a = -1, b = 1 \\ 0 \text{ otherwise} \end{cases}$$

Then we have an exact sequence

$$(1) \qquad 0 \to A \to O \overset{\delta}{\to} \omega \overset{\text{ress}}{\to} A \to 0$$

which does not change if one replaces Z by a different formal coordinate system compatible with the same formal SUSY-structure.

Proof. By definition, $\delta(z^a) = dZ \cdot az^{a-1}\zeta$, $\delta(z^b\zeta) = dZ \cdot z^b$. It follows that (1) is exact. In order to prove that ress does not depend on the choice of Z it suffices to establish a formula of the type

$$dZz^{-1}\zeta - dZ'z'^{-1}\zeta' = \delta L(Z, Z')$$

for two compatible SUSY-coordinates Z, Z'. Using the analytic identity $D_Z \log z = z^{-1}\zeta$ valid outside $z = 0$, one can guess and then easily prove a formal identity: if $z/z' \equiv 1 \bmod (z, \zeta)$ then

$$dZz^{-1}\zeta - dZ'z'^{-1}\zeta' = \delta \log zz'^{-1}.$$

The general case reduces to this one by a linear coordinate change. In fact, put

$$\zeta' = \left(\sum_{i \geq 0} a_i z^i\right)\zeta + \sum_{j \geq 0} \alpha_j z^j = f(z)\zeta + \gamma(z),$$

$$z' = \sum_{k \geq 1} b_k z^k + \sum_{\ell \geq 0} \beta_\ell z^\ell \zeta = g(z) + \beta(z)\zeta$$

Then compatibility means that $D_Z z' = \zeta' D_Z \zeta'$ (cf. e.g. [3], i.e.

$$\partial g/\partial z = f^2 - \gamma\partial\gamma/\partial z; \quad \beta = -f\gamma.$$

In particular, $b_1 = a_1^2$ + nilpotent, and we have a new compatible system $Z'' = (b_1^{-1} z', b_1^{-1/2}\zeta')$. Replacing Z' by Z'' does not change the superresidue, and $z''z^{-1} \equiv 1 \bmod (z, \zeta)$. ∎

1.5. A local computation. Now let $\sigma : S \to X$ be an S-point of X, defined by $z = z_0$, $\zeta = \zeta_0$. Consider a section ν of ω meromorphic in a neighbourhood of this point and having a pole of order $\leqslant i + 1$ at the associated divisor, i.e.

(2) $$\nu = dZ f(Z, s)(z - z_0 - \zeta\zeta_0)^{-(i+1)}, \quad f \text{ regular.}$$

Denote by $\mathrm{ress}_{(z_0, \zeta_0)}(\nu)$ the superresidue calculated, say, in the completion $\pi^{-1}(O_S)((z - z_0 - \zeta\zeta_0, \zeta - \zeta_0))$. Then we have

(3) $$\mathrm{ress}_{(z_0, \zeta_0)}(\upsilon) = \frac{1}{i!} D_Z^{2i+1}(f)|_{Z=Z_0}$$

In fact, if

$$f = \sum_{j \geqslant 0} a_j (z - z_0 - \zeta\zeta_0)^j + \sum_{k \geqslant 0} b_k (z - z_0 - \zeta\zeta_0)^k (\zeta - \zeta_0).$$

then $\mathrm{ress}_{(z_0, \zeta_0)}(\nu) = (-1)^{\tilde{b}_i} b_i$. On the other hand

$$D_Z^{2i+1} = D_Z (\partial/\partial z)^i \text{ so that } D_Z^{2i+1}(f)\,|_{Z=Z_0} = (-1)^{\tilde{b}_i} i! b_i.$$

1.6. Residue with coefficients. The invariance property of the superresidue shows that for any coherent sheaf E on S and any S-point of X there exists a map of sheaves

$$\mathrm{ress}_D : \omega \otimes \pi^* E(\infty D) \to E$$

where D is the divisor associated to this S-point. Applying this to $E = \omega$ and $D =$ superdiagonal we get two residues along $p_{1,2} : X \times_S X \to X$:

$$\mathrm{ress}^{1,2} : \omega \boxtimes \omega(\infty \Delta^S) \to O_X .$$

The following Lemma is a superanalogue of 2.1.1.1, [2].

1.6.1. Lemma. *There exists a unique map*

$$\widetilde{\mathrm{ress}} : \omega \boxtimes \omega(\infty \Delta^S) \to O_X$$

with the following properties: a) $\delta \circ \widetilde{\mathrm{ress}} = \mathrm{ress}^1 - \mathrm{ress}^2$.

b) The restriction of $\widetilde{\mathrm{ress}}$ *to sections with bounded order of pole is a differential operator along fibres of* $p_{1,2}$.

Proof. If there are two maps with such a property then their difference would be a differential operator mapping $\omega \boxtimes \omega(i\Delta^S)$ into constants $\pi^{-1}(O_S)$. Hence it must be zero. Therefore it suffices to construct $\widetilde{\mathrm{ress}}$ locally. Putting

$$v = dZ_1 \boxtimes dZ_2 f(Z_1, Z_2, s)(z_1 - z_2 - \zeta_1\zeta_2)^{-(i+1)}$$

and calculating locally via (2) and (3) we get

$$\mathrm{ress}^1(v) = -dZ_2 \frac{1}{i!} D_{Z_1}^{2i+1} f(Z_1, Z_2, s)|_{Z_1 = Z_2 = Z'}$$

$$\mathrm{ress}^2(v) = (-1)^{i+1} dZ_1 \frac{1}{i!} D_{Z_2}^{2i+1} f(Z_1, Z_2, s)|_{Z_1 = Z_2 = Z}$$

Now put

(4) $$\widetilde{\mathrm{ress}}\,(v) = \frac{1}{i!} \sum_{b=0}^{2i} (-1)^{\left[\frac{b-1}{2}\right]} D_{Z_1}^{2i-b} D_{Z_2}^{b} f(Z_1, Z_2, s)|_{Z_1 = Z_2 = Z}$$

In order to check that $\delta\, \widetilde{\mathrm{ress}}(v) = \mathrm{ress}^1(v) - \mathrm{ress}^2(v)$ denote

$$g_{ab} = D_{Z_1}^a D_{Z_2}^b f|_{Z_1 = Z_2 = Z}$$

Then

$$D_Z g_{ab} = g_{a+1,b} + (-1)^a g_{a,b+1}.$$

This leads to the cancellation of all terms in $\delta\, \widetilde{\mathrm{ress}}\,(v)$ except for the first and the last. Finally, c) is checked directly. ■

1.7. The Grothendieck - Sato description of the differential operator sheaf. Let E be a vector bundle on X. Denote by $D_{E/S}$ the sheaf of differential operators vertical (over S), acting on E on the left. There is a standard isomorphism

$$E \otimes D_{O_X/S} \otimes E^* \tilde{\to} D_{E/S}.$$

Put $E^t = \omega \otimes E^*$ and consider the sheaf $E \boxtimes E^t(\infty\, \Delta^S)$ of meromorphic sections of $E \boxtimes E^t(\infty\, \Delta^S)$, with a pole at Δ^S. For a section v of this sheaf, denote by $r(v)$ the operator $E \to E$ defined by

$$r(v)\,\ell = \mathrm{ress}^2_{\Delta^S}(v, p_2^*\,\ell)$$

where ress^2 means the superresidue along p_2 and $(v, p_2^*\,\ell)$ refers to the contraction $E \boxtimes (\omega \otimes E^* \otimes E) \to E \boxtimes \omega$.

1.7.1. Lemma. *a)* $\mathrm{Ker}\, r = E \boxtimes E^t$.

b) r defines an isomorphism

$$r : E \boxtimes E^t((i+1)\Delta^s)/E \boxtimes E^t \to D^{\leqslant 2i+1}_{E/S}$$

where $\leqslant 2i+1$ *refers to the* D_Z – *order of a differential operator (which depends only on the SUSY-structure, cf.* [3]).

Proof. a) is clear and b) follows from (3). In fact, $D_{E/S}$ is generated by powers of D_Z and End E. Odd powers are covered by r in view of (3). Combining with multiplication by $\zeta_1 - \zeta_2$ one also reconstructs even powers. ∎

1.8. **Adjoint operators.** In the previous notation, let $\langle . \rangle$: $E^t \otimes E \to \omega$ be the natural contraction map. Denote $D_{E^t/S,\ \mathrm{right}}$ the sheaf of differential operators over S acting upon E^t on the right.

1.8.1. **Lemma.** *There exists a unique ring isomorphism*

$$D_{E/S} \to D_{E^t/S,\mathrm{right}} : P \to \overleftarrow{P}$$

and a unique map

$$\{\cdot\,,\cdot\,,\cdot\} : E^t \times D_{E/S} \times E \to O_X$$

with the following properties: for any $f^t \in E^t$, $e \in E$, $P, Q \in D_{E/S}$

(5) $$\langle f^t, Pe\rangle = \langle f^t \overleftarrow{P}, e\rangle + \delta\{f^t, P, e\};$$

(6) $$P \in \mathrm{End}\, E \Rightarrow\ f^t, P, e\} = 0;$$

(7) $$\{f^t, D_Z, e\} = (-1)^{\tilde{f}+1}(dZ)^{-1}\langle f^t, e\rangle;$$

(8) $$\{f^t, QP, e\} = \{f^t \overleftarrow{Q}, P, e\} + \{f^t, Q, Pe\}$$

Proof. From (6) – (8), uniqueness of $\{f^t, P, e\}$ is clear. From (5) uniqueness of $\overleftarrow{P}$ then follows. To prove the existence, one gets an explicit formula of the type (5) in a compatible coordinate system Z, using the «integration by parts» procedure and then checks all identities. ∎

1.9. **A central extension of $\mathbf{D}_{E/S}$.** Using Lemma 1.7.1., one can describe a canonical central extension of $D_{E/S}$ considered as a Lie superalgebra (and denoted then $D^{\mathrm{Lie}}_{E/S}$) by the sheaf $H^1 = \omega_{X/S}/\delta O_X$ which is nontrivial on the fibrewise Zariski-open subsets of X. The existence of such an extension was suggested by E. Witten [9] in his discussion of [1], and it was constructed by E. Getzler [10] in the bosonic case using cyclic homology. Here it arises in a completely natural way, both in bosonic [2] and fermionic contexts. To construct it, we start with an exact sequence (7.1b):

$$(9) \qquad 0 \to E \boxtimes E^t \to E \boxtimes E^t(\infty\Delta^s) \xrightarrow{r} D_{E/S} \to 0$$

Lemma 1.8.1. allows one to define the action of $D^{\mathrm{Lie}}_{E/S}$ upon $E \boxtimes E^t(\infty\ \Delta^s)$:

$$(10) \qquad \mathrm{Lie}\, P(f) = p_1^*(P)\, v - (-1)^{\tilde{P}\tilde{v}} v p_2^*(\overleftarrow{P}),$$

for $P \in D_{E/S}$, $v \in E \boxtimes E^t(\infty\Delta^s)$.

1.9.1. Proposition. a) *$E \boxtimes E^t$ is invariant with respect to this action. The induced action upon* $\mathrm{Im}(r) = D_{E/S}$ *is the adjoint one.*

b) *Let $i_\Delta : X \to X_S \times X$ be the relative diagonal and j the composite map*

$$j : E^S \times E^t \xrightarrow{i_\Delta^*} E \otimes \omega \otimes E^* \xrightarrow{\mathrm{str}} \omega.$$

Then $D^{\mathrm{Lie}}_{E/S}\ (E \boxtimes E^t) \subset j^{-1}(\delta O)$. Therefore, factorizing (9) by $j^{-1}(\delta O)$, we get a central extension on X:

$$(11) \qquad 0 \to H^1 \to \tilde{D}^{\mathrm{Lie}}_{E/S} \to D^{\mathrm{Lie}}_{E/S} \to 0.$$

Proof. a) We must establish that for $e \in E$ we have

$$r(p_1^*(P)\, v)\,(e) - (-1)^{\tilde{P}\tilde{v}} r(vp_2^*(\overleftarrow{P}))(e) = Pr(v)(e) - (-1)^{\tilde{P}\tilde{v}} r(v)P(e).$$

The first members of both clearly coincide. The second members coincide due to the adjunction formula and the fact that $\mathrm{ress} \circ \delta = 0$.

b) Similarly, for $e \in E$, $f^t \in E^t$ we have

$$(12) \qquad i_\Delta^*(Pe \,\square\, f^t \; - (-1)^{\tilde{P}(\tilde{e}+\tilde{f})} e \boxtimes f^t\overleftarrow{P}) = (Pe) \otimes f^t - (-1)^{\tilde{P}(\tilde{e}+\tilde{f})} e \otimes (f^t\overleftarrow{P}).$$

Furthermore, $\mathrm{str}(e \otimes f^t) = (-1)^{\tilde{e}\tilde{f}} \langle f^t, e\rangle$ (one may take this as definition). Therefore, the r.h.s. of (12) after applying str becomes

$$(-1)^{\tilde{f}(\tilde{P}+\tilde{e})} \langle f^t, Pe\rangle - (-1)^{\tilde{f}(\tilde{P}+\tilde{e})} \langle f^t\overleftarrow{P}, e\rangle \in \delta O. \qquad \blacksquare$$

2. ATIYAH'S ALGEBRAS

2.1. Notation. The basic technical notion introduced in [2] is that of the Atiyah algebras (cf. also [8], ch. VIII). In order to motivate it, we recall that historically the idea of symmetry became embodied in the following chain of structures:

Lie group ⇒ Lie algebra of a Lie group ⇒ abstract Lie algebras. Similarly, the package of definitions given in [2] consists of various specializations of the following scheme:

Automorphism group of a geometric object $V \Rightarrow$
$\Rightarrow$ sheaf of infinitesimal symmetries of V (Atiyah's algebra)
$\Rightarrow$ abstract Atiyah's algebras.

The geometric object V in question may consist of a collection of manifolds, morphisms, and distributions. Therefore, its sheaf of infinitesimal symmetries is endowed with the corresponding structures. Below, we list the principal situations we have to deal with, together with the standard notation and some explanations. For an axiomatization of these «concrete» Atiyah algebras we refer to [2], § 1, and to § 3.6 below.

We work in the category of superanalytic spaces. Therefore expressions like «manifold» and «supermanifold», «vector bundle», «supervector bundle», and «locally free sheaf», etc. are used synomymously. However, involvement of a SUSY-structure is indicated in notation by a superscript s.

2.1.1. $V = X$, a manifold. Its sheaf of infinitesimal symmetries is denoted T_X, the tangent sheaf.

2.1.2. $V = (X, E)$; E is a vector bundle on X. Its sheaf of infinitesimal symmetries A_E is a Lie (super) algebra on X represented as an extension

$$0 \to \mathrm{End}\, E \to A_E \to T_X \to 0,$$

where End E is the internal endomorphism sheaf $E^*_{O_X}\otimes E$. It coincides with the sheaf of (left) differential operators on E of order $\leqslant 1$, whose symbols is identity.

2.1.3. $V = (X, S, \pi)$, where $\pi : X \to S$ is a submersion («relative manifold»). Here the sheaf of infinitesimal symmetries is denoted by T_π. It can be described as

$$T_\pi = (d\pi)^{-1}(\pi^{-1}(T_S)),\ d\pi : T_X \to \pi^*(T_S).$$

Let $T_{X/S}$ be the sheaf of vertical vector fields. Clearly, we have

$$0 \to T_{X/S} \to T_\pi \to \pi^{-1}(T_S) \to 0.$$

In other words, a vector field belongs to T_π if all vectors tangent to points of a π-fibre project onto the same vector tangent to the base space.

2.1.4. $V = (X, S, \pi, E)$, where E is a vector bundle on X. The corresponding Atiyah algebra is denoted $A_{E,\pi}$. It is embedded in the diagram

$$
(1)\qquad \begin{array}{ccccc}
\mathrm{End}\,E & = & \mathrm{End}\,E & = & \mathrm{End}\,E \\
\downarrow & & \downarrow & & \downarrow \\
A_{E/S} & \to & A_{E,\pi} & \to & A_E \\
\downarrow & & \downarrow & & \downarrow \\
T_{X/S} & \to & T_\pi & \to & T_X
\end{array}
$$

which is self-explanatory.

2.1.5. $V = (X, S, \pi,$ SUSY-structure), i.e. a SUSY-family. The sheaf of infinitesimal symmetries is denoted here by $T_\pi^S \subset T_X$. It consists of vector fields ∂ on X such that $[\partial, T^1_{X/S}] \subset T^1_{X/S}$ and fits into the exact sequence

$$0 \to T^s_{X/S} \to T^s_\pi \to \pi^{-1}(T_S) \to 0.$$

The following lemma is straightforward:

2.1.5.1. **Lemma.** *Let Z be a compatible local coordinate. Then*

$$T^s_{X/S} = \left\{ \partial_a = aD_Z^2 + \frac{(-1)^{\tilde{a}}}{2} D_Z a \cdot D_Z \mid a \in O_X \right\};\ [\partial_a, D_Z] = -\frac{1}{2} D_Z^2 a D_Z.$$

In particular,

$$T_{X/S} = T^s_{X/S} \oplus T^1_{X/S}$$

as sheaves of linear spaces. ∎

2.1.6. $V = (X, S, \pi, E,$ SUSY-structure). The corresponding Atiyah algebra will be denoted $A^s_{E,\pi}$. It fits into a diagram similar to (1):

$$
(2)\qquad \begin{array}{ccccc}
\mathrm{End}\,E & = & \mathrm{End}\,E & = & \mathrm{End}\,E \\
\downarrow & & \downarrow & & \downarrow \\
A^s_{E/S} & \xrightarrow{i} & A^s_{E,\pi} & \to & A_E \\
\downarrow & & \downarrow & & \downarrow \\
T^s_{X/S} & \to & T^s_\pi & \to & T_X
\end{array}
$$

A local description similar to that in [2] shows that $A^s_{E/S}$ is a sheaf of combined

Neveu-Schwarz-Kac-Moody superalgebras factorized by the centre.

2.2. The Ω-extension of $A^s_{E,\pi}$. We shall now describe an object $\widetilde{A}^{\bullet s}_{E,\pi}$ which embodies the information not only on the infinitesimal symmetries of V in 2.1.6 but also on its canonical central extension (which is symbolized by tilda) and which is a complex (as superscript dot indicates). The nonzero components on this complex live in dimension -2, -1, 0 and are as follows

a) $\widetilde{A}^{-2,s}_{E,\pi} = O_X$.

b) $\widetilde{A}^{0,s}_{E,\pi} = A^s_{E,\pi}$.

c) $\widetilde{A}^{-1,s}_{E,\pi}$ is defined by the following commutative diagram with exact rows on $X \underset{S}{\times} X$:

$$\begin{array}{ccccccc}
0 \to E \boxtimes E^t & \to & E \boxtimes E^t(\infty\Delta^s) & \xrightarrow{r} & D_{E/S} & \to 0 \\
 & & \cup & & & \\
\| & & E \boxtimes E^t(2\Delta^s) & & \uparrow & \\
 & & \cup & & & \\
0 \to E \boxtimes E^t & \to & B^s_E & \xrightarrow{r} & A^s_{E/S} & \to 0 \\
\downarrow \mathrm{str}\circ i^*_\Delta & & \downarrow & & \| & \\
0 \to \omega = \omega_{X/S} & \to & \widetilde{A}^{-1,s}_{E,\pi} & \xrightarrow{r} & A^s_{E/S} & \to 0
\end{array} \tag{3}$$

The first line of this diagram corresponds bo the superversion of the Grothendieck-Sato description of $D_{E/S}$ (cf. 1.7.1). The sheaf B^s_E is $r^{-1}(A^s_{E/S})$ by definition. The lower left vertical arrow is the restriction to the relative superdiagonal followed by the matrix supertrace map. The lower line is supported by the diagonal which is identified with $X : i_\Delta : X \to X \times X$.

Finally, the differentials of $\widetilde{A}^s_{E,\pi}$ can be read off the following commutative diagram whose middle column is the last line of (3):

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
O_X & \xrightarrow{\delta} & \omega & & \\
\| & & \downarrow & & \\
\widetilde{A}^{-2,s}_{E,\pi} & \to & \widetilde{A}^{-1,s}_{E,\pi} & \to & \widetilde{A}^{0,s}_{E,\pi} \\
 & & \downarrow & & \| \\
 & & A^s_{E/S} & \to & A^s_{E,\pi} \\
 & & \downarrow & & \\
 & & 0 & &
\end{array} \tag{4}$$

Our $\tilde{A}^{\bullet s}_{E,\pi}$ corresponds to ${}^{tr}A^{\bullet}_{E}$ in the pure even setting of [2].

2.3. **The brackets on** $\tilde{A}^{\bullet s}_{E,\pi}$. We shall now describe the structure of a differential graded lie superalgebra on $\tilde{A}^{\bullet s}_{E,\pi}$. We take the standard Z_2-grading upon $\tilde{A}^{0s}_{E,\pi}$ and $\tilde{A}^{-2,s}_{E,\pi}$ and the reverse one upon $\tilde{A}^{-1,s}_{E,\pi}$.

The brackets are defined as follows.

$[\ ,\ ]_{0,0}$ is the standard (super) commutator of the vector fields preserving $(X, S, \pi, \mathrm{SUSY})$.

$[\ ,\ ]_{0,-1}$ is induced by the natural action of $A^{s}_{E,\pi}$ via infinitesimal symmetries. On $A^{s}_{E/S} \subset A^{s}_{E,\pi}$, it coincides with that induced by $A^{s}_{E/S} \subset D^{\mathrm{Lie}}_{E/S}$ on $E \boxtimes E^{t}\ (\infty\Delta^{s})$ (cf. 1.9):

$$\mathrm{Lie}(P)(e \boxtimes f^{t}) = Pe \boxtimes f^{t} - (-1)^{\tilde{P}(\tilde{e}+\tilde{f})}\ e \boxtimes f^{t}\overleftarrow{P}.$$

$[\ ,\ ]_{0,-2}$ is defined as the standard action of T^{s}_{π} (which is a factor of $\tilde{A}^{0,s}_{E,\pi}$) upon $O_X = \tilde{A}^{-2,s}_{E,\pi}$.

Finally, $[\ ,\ ]_{-1,-1}$ is given by the formula

$$[e \boxtimes f^{t}, e' \boxtimes f''] = (-1)^{\tilde{e}(\tilde{f}^{t}+\tilde{e}'+\tilde{f}'^{t})}\mathrm{str}\ \widetilde{\mathrm{res}}^{E}(f^{t}e' \boxtimes f'^{t}e).$$

Here $e \boxtimes f^{t}$, $e' \boxtimes f'^{t}$ mean representatives in $E \boxtimes E^{t}(2\Delta^{s})$ of the elements of $\tilde{A}^{-1,s}_{E,\pi}$ we wish to commute (cf. (3)). The map $\widetilde{\mathrm{res}}^{E}$ is defined like in Lemma 1.6.1., only here it takes values in End E. Finally, str is the usual matrix supertrace.

2.3.1. **Lemma.** $\tilde{A}^{\bullet s}_{E,\pi}$ with these brackets is a differential graded Lie superalgebra. ∎

To verify this lemma, one must use a series of identities in order to check the following facts:

– the symmetry of $[\ ,\ ]_{-1,-1}$ justifying our choice of Z_2-gradation;

– the Jacobi identity which amounts to the verification that $\tilde{A}^{-1}$, $\tilde{A}^{-2}$ are $\tilde{A}^{0}$-modules and $[\ ,\ ]: \tilde{A}^{-1} \otimes \tilde{A}^{-1} \to \tilde{A}^{-2}$ is a morphism of $\tilde{A}^{0}$-modules;

– the compatibility of brackets with the differential $d[a, b] = [da, b] + (-1)^{\tilde{a}}[a, db]$ which sould be checked for degrees $[-2, 0]$, $[-1, -1]$ and $[-1, 0]$, only the second case being interesting.

See [2] for the bosonic case.

2.4. **Nevey-Schwarz sheaf.** Put $NS_{E/S} = \tilde{A}^{-1,s}_{E,\pi}\ /\mathrm{Im}\,\delta$. The last line of (3) provides us with an exact sequence

$$0 \to H^1 \to NS_{E/S} \to A^s_{E/S} \to 0$$

where $H^1 = \omega/\delta O_X$ on the «big» open sets which are Zariski-open along fibers of π. From Proposition 1.3.1., it follows that $NS_{E/S}$ is a central extension of $A^s_{E/S}$. A local computation as in [2] shows that $NS_{E/S}$ is a combined Neveu-Schwarz-Kac-Moody type Lie superalgebra.

3. MAIN RESULTS

3.1. **Theorem.** *In the notation of* §2.2, *we have a canonical exact sequence of sheaves on S, whose middle term inherits from* $\tilde{A}^{\bullet s}_{E,\pi}$ *the structure of a Lie superalgebra:*

$$(1) \qquad 0 \to O_S \to R^0\pi_* \tilde{A}^{\bullet s}_{E,\pi} \to T_S \to 0.$$

Comparing this with the exact sequence in 2.1.2. we see that (1) looks like the Atiyah algebra of an invertible sheaf on S. In fact it is one:

3.2. **Theorem.** *Let* λ_E *denote the Berezinian sheaf* $Ber\, R\pi_*E$ *(cf. below and* [5]). *Then there is a canonical system of isomorphisms of superalgebras*

$$I_E : R^0\pi_*\tilde{A}^{\bullet s}_{E,\pi} \tilde{\to} A_{\lambda_E}$$

verifying a list of naturalness properties (cf. [2] *and below)* ■

Consider now the case $E = \omega^j$. Put $\Lambda_j = \lambda_{\omega^j}$ and $\gamma_j = (-1)^{j-1}(2j-1)$. A.A. Voronov [5] and P. Deligne independently proved the following SUSY version of Mumford's theorem: there is a canonical isomorphism $\Lambda_j \cong \Lambda_1^{\gamma_j}$. Aconstant section μ^s_j of $\Lambda_1^{-\gamma_j}$ is called a Mumford superform. For $j = 3$, it defines the quantum measure on the moduli space (or, rather, stack) of SUSY-curves. Put $A_j = A_{\Lambda_j \otimes \Lambda_1^{-\gamma_j}}$.

3.3. **Theorem.** *Using 3.2, one can construct a canonical connection* ∇_j *on* $\Lambda_j \otimes \Lambda_1^{-\gamma_j}$. *It is flat, and its horizontal section is (proportional to)* μ_j.

Thus, writing down the differential equations $\nabla_j(\partial)(\mu_j) = 0$ for a basis of vector fields ∂ in T_S, we get a system defining μ_j up to a constant. See [2] for some concrete calculations in the bosonic case.

3.4. Proof of Theorem 3.1. In (5), $R^0\pi_*$ is the hyperdirect image of a complex of sheaves. Working in the derived category, we calculate it in the following way. Consider the following complexes (only non zero terms are written explicitly, the grading is put in brackets):

$$T_\pi^{\bullet s} : \underset{(-1)}{T^s_{X/S}} \to \underset{(0)}{T^s_\pi} \;;\; \omega^\bullet_{X/S}[2] : \underset{(-2)}{O_X} \overset{\delta}{\to} \underset{(-1)}{\omega_{X/S}}.$$

Looking at the diagrams (2), (3), (4), one sees that $\widetilde{A}^{\bullet s}_{E,\pi}$ admits a 3-step filtration $0 = \widetilde{A}^\bullet_{-3} \subset \widetilde{A}^\bullet_{-2} \subset \widetilde{A}^\bullet_{-1} \subset \widetilde{A}^\bullet_0 = \widetilde{A}^{\bullet s}_{E,\pi}$ with the following properties:

a) $\widetilde{A}^\bullet_{-2} = \omega^\bullet_{X/S}[2]$.

b) $\widetilde{A}^\bullet_0/\widetilde{A}^\bullet_{-1} = T^{\bullet S}_\pi$.

c) $\widetilde{A}^\bullet_{-1}/\widetilde{A}^\bullet_{-2}$ is isomorphic to $\underset{(-1)}{\mathrm{End}\,E} \overset{id}{\to} \underset{(0)}{\mathrm{End}\,E}$.

Since this last quotient is zero in the homotopic category, we have the distinguished triangle on X

$$\omega^\bullet_{X/S}[2] \to \widetilde{A}^{\bullet s}_{E,\pi} \to T^{\bullet s}_\pi$$

which leads to the distinguished triangle on S

$$R\pi_*\omega^\bullet_{X/S}[2] \to R\pi_*\widetilde{A}^{\bullet s}_{E,\pi} \to R\pi_* T^{\bullet s}_\pi$$

and then to the exact sequence of the cohomology sheaves

$$\ldots R^{-1}\pi_*T^{\bullet s}_\pi \to R^0\pi_*\omega^\bullet_{X/S}[2] \to R^0\pi_*\widetilde{A}^{\bullet s}_{E,\pi} \to R^0\pi_*T^{\bullet s}_\pi \to R^1\pi_*\omega^\bullet_{X/S}[2] \to \ldots$$

Clearly, $T^{\bullet s}_\pi$ is quasiisomorphic to $\pi^{-1}(T_S)$ so that

$$R^{-1}\pi_*T^{\bullet s}_\pi = 0; \qquad R^0\pi_*T^{\bullet s}_\pi = T_S.$$

Furthermore, $\omega^\bullet_{X/S}[2]$ fits into a triangle

$$\omega_{X/S}[1] \to \omega^\bullet_{X/S}[2] \to O_X[2]$$

which shows that $R^1\pi_*\omega^\bullet_{X/S}[2] = 0$ and gives an exact sequence

$$\begin{array}{cccc} R^{-1}\pi_*O_X[2] \to & R^0\pi_*\omega_{X/S}[1] \to & R^0\pi_*\omega^\bullet_{X/S}[2] \to 0 \\ \| & \| & \\ \varphi : R^1\pi_*O_X \to & R^1\pi_*\omega_{X/S}. & \end{array}$$

Clearly, φ is induced by δ. Since in the Hausdorff topology we have an exact sequence $0 \to \pi^{-1}(O_S) \to O_X \to \omega_{X/S} \to 0$, we get finally

$$R^0\pi_*\omega^\bullet_{X/S}[2] \cong \mathrm{Coker}\,\varphi \cong R^2\pi_*(\pi^{-1}(O_S)) = O_S .$$

finishing our proof. ∎

In this way, we get the trace map Tr: $R^1\pi_*\omega_{X/S} \to O_S$ which is involved in Serre duality. Note that it may fail to be an isomorphism.

3.5. Proof of Theorem 3.2. Following [2] closely, we first construct I_E directly for «sufficiently acyclic» sheaves and then show that natural compatibility properties allow us to extend this construction uniquely to all E.

First we construct a useful approximation to $\tilde{A}^{\bullet s}_{E,\pi}$ which is denoted $C^{\bullet s}_E$, together with a morphism $\beta : C^{\bullet s}_E \to \tilde{A}^{\bullet s}_{E,\pi}$:

$$\begin{array}{ccccccc}
C^{\bullet s}_E : & 0 & \to & B^s_E & \overset{i\circ r}{\to} & A^s_{E,\pi} & \\
\beta\ \downarrow & \downarrow & & \downarrow & & \| & \text{(see §2, (2) and (3))} \\
\tilde{A}^{\bullet s}_{E,\pi} : & \tilde{A}^{-2,s}_{E,\pi} & \to & \tilde{A}^{-1,s}_{E,\pi} & \to & \tilde{A}^{0,s}_{E,\pi} &
\end{array}$$

The structure of $C^{\bullet s}_E$ is clarified with the help of a distinguished triangle

$$E \boxtimes E^t[1] \to C^{\bullet s}_E \to \pi^{-1}T_S[0] \tag{2}$$

whose origin can be seen from the following commutative diagram:

$$\begin{array}{ccccc}
E \boxtimes E^t : & E \boxtimes E^t & \to & 0 & \\
\quad\downarrow \text{quis} & \downarrow & & \downarrow & \\
\operatorname{Ker}\varphi : & B^s_E & \overset{r}{\to} & A^s_{E/S} & \\
\downarrow & \| & & \downarrow i & \\
C^{\bullet s} : & B^s_E & \overset{i\circ r}{\to} & A^s_{E,\pi} & \\
 & & & \downarrow \quad \searrow & \\
\varphi\ \downarrow & \downarrow & & \downarrow \qquad T^s_\pi & \\
 & & & \downarrow \quad \swarrow & \\
 & & & T^s_\pi/T^s_{X/S} & \\
 & & & /\!/ & \\
\pi^{-1}T_S[0] : & 0 & \to & \pi^{-1}(T_S) &
\end{array}$$

(we continue identifying sheaves on X with sheaves on $X_S \times X$ supported on the diagonal).

Now we start constructing I_E.

3.5.1. Case $R\pi_* E = 0$. From Künneth and Serre duality, we get $R(\pi \times \pi)$. $(E \boxtimes E^t[1]) = 0$. From (2), one then sees that

$$R^0(\pi \times \pi)_* C_E^{\bullet s} \tilde{\to} R^0\pi_*(\pi^{-1} T_S) = T_S.$$

Now β furnishes a map

$$R^0(\pi \times \pi)_*(\beta)(C_E^{\bullet s}) = T_S \to R^0\pi_*(\tilde{A}_E^{\bullet s})$$

which defines a splitting of the exact sequence (1) as the reader can convince himself by looking into the proof of Theorem 3.1. One should also check that it commutes with brackets.

On the other hand, $\lambda_E = O_S$ canonically since E is acyclic, so that A_{λ_E} is canonically split. Hence the splitting of $R^0\pi_*(\tilde{A}_E^{\bullet s})$ we have defined gives I_E.

3.5.2. Case $R^1\pi_* E = 0$; a local section of π exists. From Künneth and Serre duality, we get in this case

$$R^i(\pi \times \pi)_*(E \boxtimes E^t[1]) = \begin{cases} \text{End}\, \pi_* E \text{ for } i = 0, \\ 0 \text{ for } i \neq 0 \end{cases}$$

Applying this to (2), we obtain an exact sequence

$$0 \to \text{End}\, \pi_* E \to R^0(\pi \times \pi)_*(C_\pi^{\bullet s}) \to T_S \to 0.$$

Comparing this with the definition of $A_{\pi_* E}$ (cf. 2.1.2), one begins to suspect that an isomorphism $J_E : R^0(\pi \times \pi)_* (C_\pi^{\bullet s}) \tilde{\to} A_{\pi_* E}$ should exist. This is in fact so, and J_E^{-1} followed $R^0(\pi \times \pi_*)(\beta)$ has as its kernel (super) traceless endomorphisms and hence defines $I_E^{-1} : A_{\lambda_E} \tilde{\to} R^0\pi_* \tilde{A}_{E,\pi}^{\bullet s}$.

To prove this, consider a Cousin resolved of $C_\pi^{\bullet s}$ constructed by means of a relative divisor T on X/S which is a linear combination of components associated to sections of π. Put $T^{(2)} = X \underset{S}{\times} T \subset X \underset{S}{\times} X$ and note that from $R^1\pi_* E = 0$, it follows that sheaves $A_{E,\pi}^s(\infty T)$, $E \times E^t(\infty T^{(2)})$, $B_E^s(\infty T^{(2)})$ are acyclic with respect to $\pi \times \pi : X_S \times X \to S$. Our resolvent is:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow & & \uparrow & & \\
B_E^s(\infty T^{(2)})/B_E^s & \overset{\rho}{\to} & A_{E,\pi}^s(\infty T)/A_{E,\pi}^s & \tilde{\to} & A_{E/S}^s(\infty T)/A_{E/S}^s \\
\uparrow & & \uparrow q & & \\
B_E^s(\infty T^{(2)}) & \to & A_{E,\pi}^s(\infty T) & & \\
\uparrow & & \uparrow & & \\
B_E^s & \to & A_{E,\pi}^s & & \\
\uparrow & & \uparrow & & \\
0 & & 0 & &
\end{array}$$

It shows that $C_{\pi}^{\bullet s}$ is quasiisomorphic to

$$\check{C}_E^{\bullet s}: \dots \to B_E^s(\infty T^{(2)}) \to A_{E,\pi}^s(\infty T) \oplus B_E^s(\infty T^{(2)})/B_E^s \overset{(i\circ r, -q)}{\to} A_{E,\pi}^s(\infty T)/A_{E,\pi}^s \to \dots$$

Using the acyclicity, one sees that $R^0\pi_*(C^{\bullet s})$ is just the middle cohomology sheaf of $R^0(\pi \times \pi_*)(\check{C}_E^{\bullet s})$, i.e.

$$R^0(\pi \times \pi)_*(C_E^{\bullet s}) = \{(\tau, b) \in \pi_* A_{E,\pi}^s(\infty T) \otimes \pi_* B_E^s(\infty T^{(2)})/B_E) \mid q(\tau) = \rho(b)\}/\text{Image of } B_E^s(\infty T^{(2)}).$$

We want to define with the help of (τ, b) a differential operator, $J_E(\tau, b)$: $E \to E$ belonging to $A_{\pi_* E}$. For $e \in \pi_* E$ we put

$$J_E(\tau, b)(e) = \tau(e) - \text{ress}_{T^{(2)}}(b, p_2^* e)$$

where $(b, p_2^* e)$ in the result of the contraction:

$$(b, p_2^*(e)) \in (E \boxtimes \omega)(2\Delta^s + \infty T^{(2)})/E \boxtimes \omega(2\Delta^s).$$

One can check that J_E is well defined on $R^0(\pi \times \pi)_* C_E^{\bullet s}$. In particular, poles cancel because $q(r) = p(b)$. Calculating J_E upon right and left term of the triangle (2) one sees that it is an isomorphism, hence $J_E : R^0(\pi \times \pi_*)(C_E^{\bullet s}) \tilde{\to} A_{\pi_* E}$. Passing from T_1 to $T_1 + T_2$ and then to T_2 one can prove that it does not depend on T. Finally, we want to check that Ker $R^0(\pi \times \pi)_*(\beta) \circ J_E^{-1}$ consists of (super) traceless endomorphisms. This follows from the fact that str: $End\ \pi_*\ E \to O_S$ is $Tr \circ R(\pi \times \pi)_*(j)$, where j is defined in 1.9.1.:

$$j : E \boxtimes E^t \overset{i_\Delta^*}{\to} E \otimes E^t \overset{\text{str}}{\to} \omega.$$

We omit the proof that I_E is a Lie algebra isomorphism: cf. [2], 2.3.4.

3.5.3. **Compatibilities.** As in [2] we now state a list of compatibilities of various I_E's defining them uniquely.

a) I_E should be compatible with base change.

b) Let $0 \to E_1 \to E \to E2 \to 0$ be an exact sequence of locally free sheaves. Then we have canonically $A_{\lambda_E} = A_{\lambda_{E_1}} \otimes A_{\lambda_{E_2}}$ defined via $I_E = \lambda_{E_1} \otimes \lambda_{E_2}$ and similarly for $R^0\pi_*(\tilde{A}_E^{\bullet s})$. The tensor product of Atiyah algebras is a special operation defined in [2], 1.1.5.4. The identification $R^0(\pi_*(\tilde{A}_{E,\pi}^{\bullet s}) = R^0\pi_*(\tilde{A}_{E_1,\pi}^{\bullet s}) \otimes (\tilde{A}_{E_2,\pi}^{\bullet s})$ is constructed in the same way as in [2], 2.3.3. cf. below. The compatibility condition is: $I_E = I_{E_1} \otimes I_{E_2}$.

c) For an exact sequence

$$0 \to E(-D) \to E \to E/E(-D) \to 0$$

where D is a divisor associated to a section of π put $E_D = R^0\pi_*(E/E(-D))$.

We have $\lambda_E = \lambda_{E(-D)} \otimes \det E_D$, so that $A_{\lambda_E} = A_{\lambda_{E(-D)}} \otimes A_{Ber E_D}$. As in [2], 2.3.2. one can define an isomorphism

$$R^0\pi_*(\widetilde{A}^{\bullet s}_{E,\pi}) = R^0\pi_*(\widetilde{A}^{\bullet s}_{E(-D),\pi}) \otimes A_{Ber E_D}$$

and require the compatibility condition

$$I_E = I_{E(-D)} \otimes id$$

where id is identity on Ber E_D.

As in [2] one easily sees that 3.5.1. together with these compatibility conditions define I_E uniquely.

Existence of I_E follows from 3.5.2. after one settles some details as in [2], 2.3.5. Namely, the construction 3.5.2. agrees with 3.5.1. and is compatible with localization. If $R^1\pi_*E_i = 0$, $i = 1, 2$, it is also compatible with b), and if $R^1\pi_*E(-D) = 0$, it is compatible with c). For a general D, choose D locally with $R^1\pi_*E(D) = 0$ and express I_E via $I_{E(D)}$. Independence of I_E is shown as in a previous argument. ■

3.6. Proof of Theorem 3.3. We must first recall some general definitions and constructions from [3].

The Atiyah algebra A_E of a vector bundle on a manifold X introduced in 2.1.2. belongs to the general class of abstract Atiyah algebras A each of which is a Lie superalgebra on X and a left O_X-module, represented as an extension (of both structures)

(3) $$0 \to R^{\mathrm{Lie}} \to A \xrightarrow{\epsilon} T_X \to 0$$

with the following properties. First, R is an associative O_X-algebra with unit (like End E) endowed with the (super) commutator $\tilde{a}\tilde{b} - (-1)^{ab} ba$. Second, T_X is endowed with the usual bracket. Third, we must have $[\alpha, ab] = [\alpha, a]b + (-1)^{\tilde{\alpha}\tilde{a}} a[\alpha, b]$ for any $\alpha \in A$ and either $a, b \in R$ or $a \in O_X$, $b \in A$; $[\alpha\ f] = \epsilon(\alpha)(f)$ for $f \in O_X \subset R$.

A connection ∇ on A is an O_X-linear section $\nabla : T_X \to A$. Its curvature c_∇: $\Lambda^2 T_X \to A$ is given by $c_\nabla(\partial_1 \wedge \partial_2) = [\nabla(\partial_1), \nabla(\partial_2)] - \nabla\ [\partial_1, \partial_2]$. A connection on A_E is the same as one on E. If $c_\nabla = 0$, ∇ is called flat, or integrable. Defining a flat connection on A is the same as giving a morphism (compatible with all the structures) $A_{O_X} \to A : \partial + f \to \nabla(\partial) + f$, $\partial \in T_X$, $f \in O_X$.

Assume that in addition to (3) we are given in associative O_X-algebra R' and a pair $\varphi = (\varphi_A, \varphi_R)$ of O_X-linear Lie superalgebra maps: $\varphi_A : A \to \mathrm{Der}\ R'$, $\varphi_R : R^{\mathrm{Lie}} \to R'^{\mathrm{Lie}}$, with the following properties: $[\varphi_R(r), r'] = [\varphi_A(r), r']$ for $r \in R$, $r' \in R'$, $\varphi_A(a)(f) = \epsilon(a)(f)$ for $a \in A$, $f \in O_X \subset R'$. We can then define a new

Atiyah algebra A' with

$$0 \to R' \to A' \xrightarrow{\epsilon'} T_X \to 0$$

and a morphism $A \to A' = \varphi_*(A)$ inducing φ_R upon R. In order to do this, we construct the semi-direct product $R' \times A$ and impose relations $(\varphi_R(r), 0) = (0, r)$ for all $r \in R$.

In particular, Atiyah algebras of the invertible sheaves on X are representable as in (3) with $R = O_X$. On this class, we can put $\varphi = (\varphi_A = \epsilon\ \ \varphi_R$ = multiplication by a $\lambda \in \mathbf{C}$) and $\varphi_*(A) := \lambda A$.

One can also use this φ_*-construction to define an extension of the construction $(A_{E_1}, A_{E_2}) \to A_{E_1 \otimes E_2}$ to the class of all Atiyah algebras.

Suppose we have (A_i, R_i, ϵ_i), $i = 1, 2$ as in (3). We can first from $R = R_1 \times R_2$ and an Atiyah algebra $B = A_1 {}_{T_X} \times A_2$, $R \subset B$. Then we can put $A_1 \otimes A_2 = \varphi_*(B)$ for $\varphi = (\varphi_B, \varphi_{A_1 \times A_2})$,

$$\varphi_B : B \to \operatorname{Der} R_1 \otimes R_2 : \varphi_B(a_1, a_2)(r_1 \otimes r_2) = [a_1, r_1] \otimes r_2 + (-1)^{\tilde{a}_2 \tilde{r}_1} \otimes [a_2, r_2]$$

$$\varphi_{R_1 \times R_2} : (R_1 \times R_2)^{\text{Lie}} \to (R_1 \otimes R_2)^{\text{Lie}} : (r_1, r_2) \to r_1 \otimes 1 + 1 \otimes r_2.$$

As in [2], one constructs a canonical isomorphism $A_{E_1} \otimes A_{E_2} \xrightarrow{\sim} A_{E_1 \otimes E_2}$ (cf. [2], 1.1). If E is invertible and $a \in \mathbf{Z}$ then $A_{E^a} = aA_E$.

We can now return to the setting of Theorem 3.3. We are going to construct a canonical isomorphism $A_{\Lambda_j} \xrightarrow{\sim} \gamma_j A_{\Lambda_1} \cong A_{\Lambda_1 \gamma_j}$. The above discussion shows that we shall get in this way an isomorphism $A_{O_S} \xrightarrow{\sim} A_j = A_{\Lambda_j \otimes \Lambda_1^{-\gamma_j}}$ i.e. an integrable connection on A_j. We can prove, that it annihilates μ_j by an extension of the argument in [2], 3.1.2 (at least, for $g \geqslant 3$) which is based upon the Deligne compacification of the stack of SUSY curves and which we omit.

In turn, we shall derive the isomorphism $A_{\Lambda_j} \xrightarrow{\sim} \gamma_j A_{\Lambda_1}$ on S from its version on X and the fact that A_{Λ_j} are certain direct images of complexes on X.

Concretely, $A_{\Lambda_j} = R^0\pi_*(\tilde{A}^{\bullet s}_{\omega, \pi})$ in view of Theorem 3.2. However, in the case $E = \omega^j$, we can devise a more economical subcomplex with the same direct image.

Namely, T^s_π acts upon ω^j via Lie derivative. This gives rise to a section

$$T^s_\pi \xrightarrow{\text{Lie}_j} A^s_{\omega^j, \pi} \to T^s_\pi.$$

Define a maximal subcomplex $A_j^{\bullet s} \subset \tilde{A}^{\bullet s}_{\omega^j, \pi}$ for which $A_j^{0, s} = \text{Lie}_j(T^s_\pi)$. Looking at the diagrams (2) and (4), §2, one sees that $A_j^{\bullet s}$ is quasiisomorphic to $\tilde{A}^{\bullet s}_{\omega^j, \pi}$. Hence we get $I_E : R^0\pi_*(A_j^{\bullet s}) \xrightarrow{\sim} A_{\Lambda_j}$. The structure of $A_j^{\bullet s}$ is displayed in the following version of (4), §2:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
\mathcal{O}_X & \overset{\delta}{\to} & \omega & & \\
\| & & \downarrow & & \\
A_j^{-2,s} & \to & A_j^{-1,s} & \to & A_j^{0,s} \\
 & & \downarrow & & \| \\
A^s_{E,\pi} & \subset & \mathrm{Lie}_j^{-1}(T^s_\pi) & \overset{\mathrm{Lie}_j}{\underset{\sim}{\to}} & T^s_\pi \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

All complexes $A_j^{\bullet s}$ belong to a generalization of the category of Atiyah algebras on X which is defined in [2], 1.2 and is called there the category of Atiyah π-algebras. The operations φ_*, x, $\otimes$, and multiplication by a complex contant can be extended to π-algebras, cf. [2], 1.2.1. Using this, one can check that it suffices to construct isomorphisms $A_j^{\bullet s} \tilde{\to} \gamma_j A_1^{\bullet s}$.

The method used in [2], 3.1 is based upon direct coordinate computations and essentially shows that the «central charges» for A_j^{-1} and A_1^{-1} differ by the factor $6j^2 - 6j + 1$. The same is true here, with γ_j instead of this factor. We restrict ourselves to listing some useful formulas needed to check it.

a) Let $\partial_a = aD_Z^2 + \dfrac{(-1)^{\tilde{a}}}{2} D_Z a \cdot D_Z \in T^s_{X/S}$, as in 2.1.5.1. Then

$$\mathrm{Lie}_j \cdot (\partial_a)\,(dZ)^j = \frac{j}{2} D_Z^2 a \cdot (dZ)^j.$$

b) Let $v_a \in E \boxtimes E^t(\infty\Delta^s)$ for $E = \omega^j$ be an element such that $r(v_a) = \mathrm{Lie}_j(\partial_a)$. Then

$$v_a = -\left[\frac{a(Z_1)(\zeta_2 - \zeta_1)}{(z_2 - z_1 - \zeta_2\zeta_1)^2} + \frac{(-1)^{\tilde{a}} D_{Z_1} a + j(\zeta_2 - \zeta_1) D_{Z_1}^2 a}{(z_2 - z_1 - \zeta_2\zeta_1)}\right] dZ_1^j dZ_2^{1-j} \bmod E \boxtimes E^t$$

c) $[\partial_a, \partial_b] = \partial_{\{a,b\}}$, where $\{a, b\} = aD_Z^2 b + \dfrac{(-1)^{\tilde{a}}}{2} D_Z a D_Z b - D_Z^2 a \cdot b$

d) $\mathrm{Lie}_j(\partial_a) = (-1)^{\tilde{a}j} dZ^j \otimes \left[aD_Z^2 + \dfrac{(-1)^{\tilde{a}}}{2} D_Z a \cdot D_Z + \dfrac{j}{2} D_Z^2 a\right] \otimes (dZ)^{-j}$. ■

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